

FUZZY n -ARY GROUPS AS A GENERALIZATION OF ROSENFELD'S FUZZY GROUPS

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ABSTRACT. The notion of an n -ary group is a natural generalization of the notion of a group and has many applications in different branches. In this paper, the notion of (normal) fuzzy n -ary subgroup of an n -ary group is introduced and some related properties are investigated. Characterizations of fuzzy n -ary subgroups are given.

1. PRELIMINARIES

A nonempty set G together with one n -ary operation $f : G^n \longrightarrow G$, where $n \geq 2$, is called an n -ary groupoid and is denoted by (G, f) . According to the general convention used in the theory of n -ary groupoids the sequence of elements x_i, x_{i+1}, \dots, x_j is denoted by x_i^j . In the case $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \dots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_1, \dots, x_n) = f(x_1^n)$ and

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_t, x_{i+t+1}, \dots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n -ary groupoid (G, f) is called (i, j) -associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \quad (1)$$

holds for all $x_1, \dots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i < j \leq n$, then we say that the operation f is *associative* and (G, f) is called an *n -ary semigroup*. It is clear that an n -ary groupoid is associative if and only if it is $(1, j)$ -associative for all $j = 2, \dots, n$. In the binary case (i.e. for $n = 2$) it is a usual semigroup.

If for all $x_0, x_1, \dots, x_n \in G$ and fixed $i \in \{1, \dots, n\}$ there exists an element $z \in G$ such that

$$f(x_1^{i-1}, z, x_{i+1}^n) = x_0, \quad (2)$$

then we say that this equation is *i -solvable* or *solvable at the place i* . If this solution is unique, then we say that (2) is *uniquely i -solvable*.

An n -ary groupoid (G, f) uniquely solvable for all $i = 1, \dots, n$ is called an *n -ary quasigroup*. An associative n -ary quasigroup is called an *n -ary group*. It is clear that for $n = 2$ we obtain a usual group.

Note by the way that in many papers n -ary semigroups (n -ary groups) are called n -semigroups (n -groups, respectively). Moreover, in many papers, where the arity of the basic operation does not play a crucial role, we can find the term a *polyadic semigroup* (*polyadic group*) (cf. [19]).

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Now such and similar n -ary systems have many applications in different branches. For example, in the theory of automata [13] n -ary semigroups and n -ary groups are used, some n -ary groupoids are applied in the theory of quantum groups [17]. Different applications of ternary structures in physics are described by R. Kerner in [16]. In physics there are used also such structures as n -ary Filippov algebras (see [18]) and n -Lie algebras (see [21]).

The idea of investigations of such groups seems to be going back to E. Kasner's lecture [15] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first important paper concerning the theory of n -ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [2]). In this paper Dörnte observed that any n -ary groupoid (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ \dots \circ x_n \circ b$, where (G, \circ) is a group and b belongs to the center of this group, is an n -ary group but for every $n > 2$ there are n -ary groups which are not of this form. In the first case we say that an n -ary group (G, f) is *b-derived* (or *derived* if b is the identity of (G, \circ)) from the group (G, \circ) , in the second – *irreducible*. Moreover, in some n -ary groups there exists an element e (called an *n-ary neutral element*) such that

$$f\left(\begin{smallmatrix} (i-1) \\ e \end{smallmatrix}, x, \begin{smallmatrix} (n-i) \\ e \end{smallmatrix}\right) = x \quad (3)$$

holds for all $x \in G$ and for all $i = 1, \dots, n$. It is interesting that each n -ary group (G, f) containing a neutral element are derived from a binary group (G, \circ) , where $x \circ y = f(x, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, y)$ (cf. [2]). On the other hand, there are n -ary groups with two, three and more neutral elements. All n -ary groups with this property are derived from the commutative group of the exponent $k|(n-1)$.

It is worthwhile to note that in the definition of an n -ary group, under the assumption of the associativity of f , it suffices only to postulate the existence of a solution of (2) at the places $i = 1$ and $i = n$ or at one place i other than 1 and n . Then one can prove the uniqueness of the solution of (2) for all $i = 1, \dots, n$ (cf. [19], p. 213¹⁷). Some other definitions of n -ary groups one can find in [3] and [10].

In an n -ary group the role of the inverse element plays the so-called *skew element*, i.e., an element \bar{x} such that

$$f\left(\begin{smallmatrix} (n-1) \\ x \end{smallmatrix}, \bar{x}\right) = x. \quad (4)$$

It is uniquely determined, but $\bar{x} = \bar{y}$ do not implies $x = y$, in general. Moreover, there are n -ary groups in which one element is skew to all (cf. [5]). So, in general, the skew element to \bar{x} is not equal to x , but in ternary ($n = 3$) groups we have $\bar{\bar{x}} = x$ for all $x \in G$. For some elements of n -ary groups we have $\bar{x} = x$. Such elements are called *idempotents*. An n -ary group in which elements are idempotents is called *idempotent*. There are n -ary groups without idempotents. A simple example of n -ary groups with only one idempotent are *unipotent* n -ary groups described in [6]. In these groups there exists an element θ such that $f(x_1^n) = \theta$ holds for all $x_1^n \in G$.

Note that in all n -ary groups the following two identities

$$f\left(y, \begin{smallmatrix} (i-2) \\ x \end{smallmatrix}, \bar{x}, \begin{smallmatrix} (n-i) \\ x \end{smallmatrix}\right) = f\left(\begin{smallmatrix} (n-j) \\ x \end{smallmatrix}, \bar{x}, \begin{smallmatrix} (j-2) \\ x \end{smallmatrix}, y\right) = y \quad (5)$$

are satisfied for all $2 \leq i, j \leq n$ (cf. [2]).

A nonempty subset H of an n -ary group (G, f) is an *n-ary subgroup* if (H, f) is an n -ary group, i.e., if it is closed under the operation f and $x \in H$ implies $\bar{x} \in H$ (cf. [2]). The intersection of two subgroups may be the empty set. Moreover, there

are n -ary groups which are set theoretic union of disjoint isomorphic subgroups (cf. for example [4]).

Fixing in an n -ary operation f , where $n \geq 3$, the elements a_2^{n-1} we obtain the new binary operation $x \diamond y = f(x, a_2^{n-2}, y)$. If (G, f) is an n -ary group then (G, \diamond) is a group. Choosing different elements a_2^{n-2} we obtain different groups. All these groups are isomorphic [12]. So, we can consider only groups of the form $ret_a(G, f) = (G, \circ)$, where $x \circ y = f(x, \overset{(n-2)}{a}, y)$. In this group $e = \bar{a}$, $x^{-1} = f(\bar{a}, \overset{(n-3)}{x}, \bar{x}, \bar{a})$.

Subgroups of (G, f) are not subgroups of (G, \circ) , in general.

In the theory of n -ary groups a very important role plays the following theorem firstly proved by M. Hosszú [14] (see also [11]).

Theorem 1.1. *For any n -ary group (G, f) there exist a group (G, \circ) , its automorphism φ and an element $b \in G$ such that*

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \dots \circ \varphi^{n-1}(x_n) \circ b \quad (6)$$

holds for all $x_1^n \in G$.

One can proved (see for example [11]) that in this theorem $(G, \circ) = ret_a(G, f)$, $\varphi(x) = f(\bar{a}, x, \overset{(n-2)}{a})$, $b = f(\bar{a}, \dots, \bar{a})$, where a is an arbitrary element of G . The above representation is unique up to isomorphism.

Since, as it is not difficult to see,

$$\varphi^{n-1}(x) \circ b = b \circ x$$

the identity (6) can be written in more useful form

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \dots \circ \varphi^{n-2}(x_n) \circ b \circ x_n. \quad (7)$$

2. FUZZY n -ARY SUBGROUPS

Any function $\mu : G \rightarrow [0, 1]$ is called a *fuzzy subset* of G . The set of all values of μ is denoted by $\text{Im}(\mu)$. If for every $S \subseteq G$, there exists $x_0 \in S$ such that $\mu(x_0) = \sup\{\mu(x) \mid x \in S\}$ then we say that μ has *sup-property*.

For usual groups A. Rosenfeld defined [20] fuzzy subgroups in the following way:

Definition 2.1. A fuzzy subset μ defined on a group (G, \cdot) is called a *fuzzy subgroup* if

- 1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
- 2) $\mu(x^{-1}) \geq \mu(x)$

holds for all $x, y \in G$.

In fact we have $\mu(x^{-1}) = \mu(x)$ because $(x^{-1})^{-1} = x$ for every $x \in G$. Moreover, from the above definition we can deduce that $\mu(e) \geq \mu(x)$ for every $x \in G$.

Proposition 2.2. [20] *A fuzzy subset μ on a group (G, \cdot) is a fuzzy subgroup if and only if each nonempty level subset $\mu_t = \{x \in G \mid \mu(x) \geq t\}$ is a subgroup of (G, \cdot) .*

The above definition can be extended to n -ary case in the following way (cf. [8]):

Definition 2.3. Let (G, f) be an n -ary group. A fuzzy subset of G is called a *fuzzy n -ary subgroup* of (G, f) if the following axioms hold:

- (i) $\mu(f(x_1^n)) \geq \min\{\mu(x_1), \dots, \mu(x_n)\}$ for all $x_1^n \in G$,
- (ii) $\mu(\bar{x}) \geq \mu(x)$ for all $x \in G$.

Note that for $n = 3$ the second condition of the Definition 2.3 can be replaced by the condition

$$(iii) \quad \mu(\bar{x}) = \mu(x) \text{ for all } x \in G,$$

because in this case $n = 3$ we have $\bar{\bar{x}} = x$ for all $x \in G$ (cf. [2]). These two conditions are equivalent for all n -ary groups in which for every $x \in G$ there exists a natural number k such that $\bar{x}^{(k)} = x$, where $\bar{x}^{(k)}$ denotes the element skew to $\bar{x}^{(k-1)}$ and $\bar{x}^{(0)} = x$. But, as it was observed in [8], there are fuzzy n -ary subgroups in which $\mu(\bar{x}) > \mu(x)$ for some $x \in G$.

Example 2.4. Let (\mathbb{Z}_4, f) be a 4-ary group derived from the additive group \mathbb{Z}_4 . It is not difficult to see that the map μ defined by $\mu(0) = 1$ and $\mu(x) = 0.5$ for all $x \neq 0$ is a fuzzy 4-ary subgroup in which for $x = 2$ we have $\bar{x} = 0$ and $\mu(\bar{x}) > \mu(x)$.

Proposition 2.5. *Any n -ary subgroup of (G, f) can be realized as a level subset of some fuzzy n -ary subgroup of G .*

Proof. Let H be an n -ary subgroup of a given n -ary group (G, f) and let μ_H be a fuzzy subset of G defined by

$$\mu_H(x) = \begin{cases} t & \text{if } x \in H \\ s & \text{if } x \notin H \end{cases}$$

where $0 \leq s < t \leq 1$ is fixed. It is not difficult to see that μ is a fuzzy n -ary subgroup of G such that $\mu_t = H$. \square

Corollary 2.6. *The characteristic function of a nonempty subset of an n -ary group (G, f) is a fuzzy n -ary subgroup of G if and only if A is an n -ary subgroup of G .*

Theorem 2.7. *A fuzzy subset μ on an n -ary group (G, f) is a fuzzy n -ary subgroup if and only if each its nonempty level subset is an n -ary subgroup of (G, \cdot) .*

Proof. Let μ be a fuzzy n -ary subgroup of an n -ary group (G, f) . If $x_1^n \in \mu_t$ for some $t \in [0, 1]$, then $\mu(x_i) \geq t$ for all $i = 1, 2, \dots, n$. Thus

$$\mu(f(x_1^n)) \geq \min\{\mu(x_1), \dots, \mu(x_n)\} \geq t,$$

which implies $f(x_1^n) \in \mu_t$. Moreover, for $x \in \mu_t$ from $\mu(\bar{x}) \geq \mu(x) \geq t$ it follows $\bar{x} \in \mu_t$. So, μ_t is an n -ary subgroup of (G, f) .

Conversely, assume that every nonempty level subset μ_t is an n -ary subgroup of (G, f) . Let $t_0 = \min\{\mu(x_1), \dots, \mu(x_n)\}$ for some $x_1^n \in G$. Then obviously $x_1^n \in \mu_{t_0}$, consequently, $f(x_1^n) \in \mu_{t_0}$. Thus

$$\mu(f(x_1^n)) \geq t_0 = \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

Now let $x \in \mu_t$. Then $\mu(x) = t_0 \geq t$, i.e., $x \in \mu_{t_0}$. Since, by the assumption, every nonempty level set of μ is an n -ary subgroup, $\bar{x} \in \mu_{t_0}$. Whence $\mu(\bar{x}) \geq t_0 = \mu(x)$.

In this way the conditions of Definition 2.3 are verified. This completes the proof. \square

Using this theorem we can prove the another characterization of fuzzy n -ary subgroups.

Theorem 2.8. *A fuzzy subset μ on an n -ary group (G, f) is a fuzzy n -ary subgroup if and only if for all $i = 1, 2, \dots, n$ and all $x_1^n \in G$ it satisfies the following two conditions*

$$(i) \quad \mu(f(x_1^n)) \geq \min\{\mu(x_1), \dots, \mu(x_n)\},$$

$$(ii) \quad \mu(x_i) \geq \min\{\mu(x_1), \dots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), \dots, \mu(x_n)\}.$$

Proof. Assume that μ is a fuzzy n -ary subgroup of (G, f) . Similarly as in the proof of Theorem 2.7 we can prove that each nonempty level subset μ_t is closed under the operation f , i.e., $x_1^n \in \mu_t$ implies $f(x_1^n) \in \mu_t$.

Now let $x_0, x_1^{i-1}, x_{i+1}^n$, where $x_0 = f(x_1^{i-1}, z, x_{i+1}^n)$ for some $i = 1, 2, \dots, n$ and $z \in G$, be in μ_t . Then, according to (ii), $\mu(z) \geq t$, which proves $z \in \mu_t$. So, the equation (2) has a solution $z \in \mu_t$. This means that each nonempty μ_t is an n -ary subgroup.

Conversely, if all nonempty level subsets of μ are n -ary subgroups, than, similarly as in the previous proof, we can see that the condition (i) is satisfied. Moreover, if for $x_1^n \in G$ we have

$$t = \min\{\mu(x_1), \dots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), \dots, \mu(x_n)\},$$

then $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in \mu_t$. Whence, according to the definition of an n -ary group, we conclude $x_i \in \mu_t$. Thus $\mu(x_i) \geq t$. This proves (ii). \square

Corollary 2.9. *A fuzzy subset μ defined on a group (G, \cdot) is a fuzzy subgroup if and only if*

- 1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\},$
- 2) $\mu(x) \geq \min\{\mu(y), \mu(xy)\},$
- 3) $\mu(y) \geq \min\{\mu(x), \mu(xy)\}$

holds for all $x, y \in G$.

Theorem 2.10. *Let μ be a fuzzy n -ary subgroup of (G, f) . If there exists an element $a \in G$ such that $\mu(a) \geq \mu(x)$ for every $x \in G$, then μ is a fuzzy subgroup of a group $\text{ret}_a(G, f)$.*

Proof. Indeed,

$$\mu(x \circ y) = \mu(f(x, \overset{(n-2)}{a}, y)) \geq \min\{\mu(x), \mu(a), \mu(y)\} = \min\{\mu(x), \mu(y)\}$$

and

$$\mu(x^{-1}) = \mu(f(\overline{a}, \overset{(n-3)}{x}, \overline{x}, \overline{a})) \geq \min\{\mu(x), \mu(\overline{x}), \mu(a), \mu(\overline{a})\} = \mu(x),$$

which completes the proof. \square

The assumption that $\mu(a) \geq \mu(x)$ cannot be omitted.

Example 2.11. Let (\mathbb{Z}_4, f) be a ternary group from Example 2.4. Then a fuzzy set μ defined by $\mu(0) = 1, \mu(2) = 0.5, \mu(1) = \mu(3) = 0.3$ is a fuzzy ternary subgroup of (\mathbb{Z}_4, f) . (This fact follows also from our Theorem 3.5 because $\{0\}$ and $\{0, 2\}$ are subgroups of (\mathbb{Z}_4, f) .) For $\text{ret}_1(\mathbb{Z}_4, f)$ we have $\mu(2 \circ 2) = \mu(f(2, 1, 2)) = \mu(1) = 0.3 < \min\{\mu(2), \mu(2)\} = 0.5$. So, the assumption $\mu(a) \geq \mu(x)$ cannot be omitted.

Theorem 2.12. *Let (G, f) be an n -ary group. If μ is a fuzzy subgroup of a group $\text{ret}_a(G, f)$ and $\mu(a) \geq \mu(x)$ for all $x \in G$, then μ is a fuzzy n -ary group of (G, f) .*

Proof. According to Theorem 1.1 any n -ary group can be presented in the form (6), where $(G, \circ) = \text{ret}_a(G, f)$, $\varphi(x) = f(\overline{a}, x, \overset{(n-2)}{a})$ and $b = f(\overline{a}, \dots, \overline{a})$. Obviously,

$$\mu(\varphi(x)) = \mu(f(\overline{a}, x, \overset{(n-2)}{a})) \geq \min\{\mu(\overline{a}), \mu(x), \mu(a)\} = \mu(x),$$

$$\mu(\varphi^2(x)) = \mu(f(\overline{a}, \varphi(x), \overset{(n-2)}{a})) \geq \min\{\mu(\overline{a}), \mu(\varphi(x)), \mu(a)\} = \mu(\varphi(x)) \geq \mu(x).$$

Consequently, $\mu(\varphi^k(x)) \geq \mu(x)$ for all $x \in G$ and $k \in \mathbb{N}$.

Similarly

$$\mu(b) = \mu(f(\bar{a}, \dots, \bar{a})) \geq \mu(\bar{a}) \geq \mu(x)$$

for every $x \in G$.

Therefore

$$\begin{aligned} \mu(f(x_1^n)) &= \mu(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \dots \circ \varphi^{n-1}(x_n) \circ b) \\ &\geq \min\{\mu(x_1), \mu(\varphi(x_2)), \mu(\varphi^2(x_3)), \dots, \mu(\varphi^{n-1}(x_n)), \mu(b)\} \\ &\geq \min\{\mu(x_1), \mu(x_2), \mu(x_3), \dots, \mu(x_n), \mu(b)\} \\ &\geq \min\{\mu(x_1), \mu(x_2), \mu(x_3), \dots, \mu(x_n)\}, \end{aligned}$$

which proves that the first condition of the Definition 2.3 is satisfied.

To prove the second condition observe that from (4) and (7) it follows

$$\bar{x} = (\varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b)^{-1}.$$

Thus

$$\begin{aligned} \mu(\bar{x}) &= \mu\left((\varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b)^{-1}\right) \\ &\geq \mu(\varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi^2(x)), \dots, \mu(\varphi^{n-2}(x)), \mu(b)\} \\ &\geq \min\{\mu(x), \mu(b)\} = \mu(x), \end{aligned}$$

which completes the proof. \square

Corollary 2.13. *If (G, f) is a ternary group, then any fuzzy subgroup of $\text{ret}_a(G, f)$ is a fuzzy ternary subgroup of (G, f) .*

Proof. Since \bar{a} is a neutral element of a group $\text{ret}_a(G, f)$ then $\mu(\bar{a}) \geq \mu(x)$ for all $x \in G$. Thus $\mu(\bar{a}) \geq \mu(a)$. But in ternary group $\bar{a} = a$ for any $a \in G$, whence $\mu(a) = \mu(\bar{a}) \geq \mu(a)$. So, $\mu(a) = \mu(\bar{a}) \geq \mu(x)$ for all $x \in G$. This means that the assumptions of Theorem 2.12 are satisfied. \square

Example 2.14. Consider the ternary group (G, f) , derived from the additive group \mathbb{Z}_4 . Let μ be a fuzzy subgroup of the group $\text{ret}_1(G, f)$ induced by subgroups $S_1 = \{11\}$, $S_2 = \{5, 11\}$ and $S_3 = \{1, 3, 5, 7, 9, 11\}$, i.e., let $\mu(11) = t_1$, $\mu(5) = t_2$, $\mu(1) = \mu(3) = \mu(7) = \mu(9) = t_3$ and $\mu(x) = t_4$ for $x \notin S_3$, where $0 \leq t_4 < t_3 < t_2 < t_1 \leq 1$. Then $\mu(\bar{5}) = \mu(7) = t_3 < t_2 = \mu(5)$, which means that μ is not a fuzzy ternary subgroup of $\mu(11) = t_1$.

From the above example it follows that:

- (1) There are fuzzy subgroups of $\text{ret}_a(G, f)$ which are not fuzzy n -ary subgroup of (G, f) .
- (2) In Theorem 2.12 the assumption $\mu(a) \geq \mu(x)$ cannot be omitted. In the above example we have $\mu(1) = t_3 < t_2 = \mu(5)$.
- (3) The assumption $\mu(a) \geq \mu(x)$ cannot be replaced by the natural assumption $\mu(\bar{a}) \geq \mu(x)$ (\bar{a} is the identity of $\text{ret}_a(G, f)$). In the above example $\bar{1} = 11$ and $\mu(11) \geq \mu(x)$ for all $x \in \mathbb{Z}_{12}$ but μ is not a fuzzy n -ary subgroup of (\mathbb{Z}_{12}, f) .

Theorem 2.15. *Let (G, f) be an n -ary group b -derived from the group (G, \circ) . Any fuzzy subgroup μ of (G, \circ) such that $\mu(b) \geq \mu(x)$ for every $x \in G$ is a fuzzy n -ary group of (G, f) .*

Proof. The first condition of the Definition 2.3 is obvious. To prove the second observe that in an n -ary group (G, f) b -derived from the group (G, \circ)

$$\bar{x} = (x^{n-2} \circ b)^{-1},$$

where x^{n-2} is the power of x in (G, \circ) .

Therefore

$$\mu(\bar{x}) = \mu((x^{n-2} \circ b)^{-1}) \geq \mu(x^{n-2} \circ b) \geq \min\{\mu(x), \mu(b)\} = \mu(x)$$

for all $x \in G$. The proof is complete. \square

Corollary 2.16. *Any fuzzy subgroup of a group (G, \circ) is a fuzzy n -ary subgroup of an n -ary group derived from (G, \circ) .*

Proof. If an n -ary group (G, f) is derived from the group (G, \circ) then $b = e$ and $\mu(e) \geq \mu(x)$ for all $x \in G$. \square

3. CHARACTERIZATIONS OF FUZZY n -ARY SUBGROUPS

Lemma 3.1. *Two level subsets μ_s, μ_t ($s < t$) of a fuzzy n -ary subgroup μ of G are equal if and only if there is no $x \in G$ such that $s \leq \mu(x) < t$.*

Proof. Let $\mu_s = \mu_t$ for some $s < t$. If there exists $x \in G$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s , which is a contradiction. Conversely assume that there is no $x \in G$ such that $s \leq \mu(x) < t$. If $x \in \mu_s$, then $\mu(x) \geq s$, and so $\mu(x) \geq t$, because $\mu(x)$ does not lie between s and t . Thus $x \in \mu_t$, which gives $\mu_s \subseteq \mu_t$. The converse inclusion is obvious since $s < t$. Therefore $\mu_s = \mu_t$. \square

Proposition 3.2. *Let μ and λ be two fuzzy n -ary subgroups of G with the same family of levels. If $\text{Im}(\mu) = \{t_1, \dots, t_m\}$ and $\text{Im}(\lambda) = \{s_1, \dots, s_p\}$, where $t_1 > t_2 > \dots > t_m$ and $s_1 > s_2 > \dots > s_p$, then*

- (i) $m = p$,
- (ii) $\mu_{t_i} = \lambda_{s_i}$ for $i = 1, \dots, m$,
- (iii) if $\mu(x) = t_i$, then $\lambda(x) = s_i$ for $x \in G$ and $i = 1, \dots, m$.

Proof. (i) and (ii) are obvious. To prove (iii) consider $x \in G$ such that $\mu(x) = t_i$. If $\lambda(x) = s_j$ then $s_j \geq s_i$, i.e., $\lambda_{s_j} \subseteq \lambda_{s_i}$. Since $x \in \lambda_{s_j} = \mu_{t_j}$, we obtain $t_i = \mu(x) \geq t_j$, which gives $\mu_{t_i} \subseteq \mu_{t_j}$. Consequently, $\lambda_{s_i} = \mu_{t_i} \subseteq \mu_{t_j} = \lambda_{s_j}$. Thus $\lambda_{s_i} = \lambda_{s_j}$. Lemma 3.1 completes the proof. \square

Theorem 3.3. *Let μ and λ be two fuzzy n -ary subgroups of G with the same family of levels. Then $\mu = \lambda$ if and only if $\text{Im}(\mu) = \text{Im}(\lambda)$.*

Proof. Let $\text{Im}(\mu) = \text{Im}(\lambda) = \{s_1, \dots, s_n\}$ and $s_1 > \dots > s_n$. By Proposition 3.2 for each $x \in G$ there exists s_i such that $\mu(x) = s_i = \lambda(x)$. Thus $\mu(x) = \lambda(x)$ for all $x \in G$, which gives $\mu = \lambda$. \square

Theorem 3.4. *Let $\{H_i \mid i \in I\}$, where $I \subseteq [0, 1]$, be a collection of n -ary subgroups of G such that*

- (i) $G = \bigcup_{i \in I} H_i$,
- (ii) $i > j \iff H_i \subset H_j$ for all $i, j \in I$.

Then μ defined by $\mu(x) = \sup\{i \in I \mid x \in H_i\}$ is a fuzzy n -ary subgroup of G .

Proof. By Theorem 2.7, it is sufficient to show that every nonempty level μ_k is an n -ary subgroup of G . Let μ_k be non-empty for some fixed $k \in [0, 1]$. Then

$$\begin{aligned} k &= \sup\{i \in I \mid i < k\} = \sup\{i \in I \mid H_k \subset H_i\} \\ \text{or} \\ k &\neq \sup\{i \in I \mid i < k\} = \sup\{i \in I \mid H_k \subset H_i\}. \end{aligned}$$

In the first case we have $\mu_k = \bigcap_{i < k} H_i$, because

$$x \in \mu_k \iff x \in H_i \text{ for all } i < k \iff x \in \bigcap_{i < k} H_i.$$

In the second case, there exists $\varepsilon > 0$ such that $(k - \varepsilon, i) \cap I = \emptyset$. In this case $\mu_k = \bigcup_{i \geq k} H_i$. Indeed, if $x \in \bigcup_{i \geq k} H_i$, then $x \in H_i$ for some $i \geq k$, which gives $\mu(x) \geq i \geq k$. Thus $x \in \mu_k$, i.e., $\bigcup_{i \geq k} H_i \subseteq \mu_k$.

Conversely, if $x \notin \bigcup_{i \geq k} H_i$, then $x \notin H_i$ for all $i \geq k$, which implies $x \notin H_i$ for all $i > k - \varepsilon$, i.e., if $x \in H_i$ then $i \leq k - \varepsilon$. Thus $\mu(x) \leq k - \varepsilon$. Therefore $x \notin \mu_k$. Hence $\mu_k \subseteq \bigcup_{i \geq k} H_i$, and in the consequence $\mu_k = \bigcup_{i \geq k} H_i$. This completes the proof. \square

Theorem 3.5. *Let μ be a fuzzy set in G and let $\text{Im}(\mu) = \{t_0, t_1, \dots, t_m\}$, where $t_0 > t_1 > \dots > t_m$. If $H_0 \subset H_1 \subset \dots \subset H_m = G$ are n -ary subgroups of G such that $\mu(H_k \setminus H_{k-1}) = t_k$ for $k = 0, 1, \dots, m$, where $H_{-1} = \emptyset$, then μ is a fuzzy n -ary subgroup.*

Proof. For any fixed elements $x_1, \dots, x_n \in G$ there exists only one $k = 0, 1, \dots, m$ such that $f(x_1^n)$ belongs to $H_k \setminus H_{k-1}$. If all x_1, \dots, x_n belongs to H_k , then at least one lies in $H_k \setminus H_{k-1}$ because in the opposite case $x_1^n \in H_{k-1}$ implies $f(x_1^n) \in H_{k-1}$ which is a contradiction. So, in this case

$$\mu(f(x_1^n)) = t_k = \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

If x_1, \dots, x_n are not in H_k , then at least one of them belongs to some $H_p \setminus H_{p-1}$, where $p > k$. Then

$$\mu(f(x_1^n)) = t_k \geq t_p \geq \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

This means that the first condition of the Definition 2.3 is satisfied in any case.

The condition is obvious since $x \in H_k \setminus H_{k-1}$ implies $\bar{x} \in H_k$. Thus $\mu(\bar{x}) \geq \mu(x)$. \square

Corollary 3.6. *Let μ be a fuzzy set in G with $\text{Im}(\mu) = \{t_0, t_1, \dots, t_m\}$, where $t_0 > t_1 > \dots > t_m$. If $H_0 \subset H_1 \subset \dots \subset H_m = G$ are n -ary subgroups of G such that $\mu(H_k) \geq t_k$ for $k = 0, 1, \dots, m$, then μ is a fuzzy n -ary subgroup in G .*

Corollary 3.7. *If $\text{Im}(\mu) = \{t_0, t_1, \dots, t_m\}$, where $t_0 > t_1 > \dots > t_m$, is the image of a fuzzy n -ary subgroup μ in G , then all level subsets μ_{t_k} are n -ary subgroups of G such that $\mu(\mu_{t_0}) = t_0$ and $\mu(\mu_{t_k} \setminus \mu_{t_{k-1}}) = t_k$ for $k = 1, 2, \dots, m$.*

Proof. By Theorem 2.7 all level subsets μ_{t_k} are n -ary subgroups. Clearly $\mu(\mu_{t_0}) = t_0$. Since $\mu(\mu_{t_1}) \geq t_1$, then $\mu(x) = t_0$ for $x \in \mu_{t_0}$ and $\mu(x) = t_1$ for $x \in \mu_{t_0} \setminus \mu_{t_1}$. Repeating this procedure, we conclude that $\mu(\mu_{t_k} \setminus \mu_{t_{k-1}}) = t_k$ for $k = 1, 2, \dots, m$. \square

Theorem 3.8. *Let (G, f) be a unipotent n -ary group. If μ is a fuzzy n -ary subgroup of G with the image $\text{Im}(\mu) = \{t_i : i \in I\}$ and $\Omega = \{\mu_t : t \in \text{Im}(\mu)\}$, then*

- (i) there exists a unique $t_0 \in \text{Im}(\mu)$ such that $t_0 \geq t$ for all $t \in \text{Im}(\mu)$,
- (ii) G is the set-theoretic union of all $\mu_t \in \Omega$,
- (iii) the members of Ω form a chain,
- (iv) Ω contains all level n -ary subgroups of μ if and only if μ attains its infimum on all n -ary subgroups of G .

Proof. (i) From the fact that in a unipotent n -ary group (G, f) there exists an element θ such that $f(x_1^n) = \theta$ for all $x_1^n \in G$ it follows

$$t_0 = \mu(c) = \mu(f(x_1^n)) \geq \min\{\mu(x_1), \dots, \mu(x_n)\}$$

for all $x_1^n \in G$. Whence we conclude (i).

(ii) If $x \in G$, then $t_x = \mu(x) \in \text{Im}(\mu)$. This implies $x \in \mu_{t_x} \subseteq \bigcup \mu_t \subseteq G$, where $t \in \text{Im}(\mu)$, which proves (ii).

(iii) Since $\mu_{t_i} \subseteq \mu_{t_j} \iff t_i \geq t_j$ for $i, j \in I$, then the family Ω is totally ordered by inclusion.

(iv) Suppose that Ω contains all level n -ary subgroups of μ . Let S be an n -ary subgroup of G . If μ is constant on S , then we are done. Assume that μ is not constant on S . We consider two cases: (1) $S = G$ and (2) $S \subset G$. For $S = G$ let $\beta = \inf \text{Im}(\mu)$. Then $\beta \leq t \in \text{Im}(\mu)$, i.e. $\mu_\beta \supseteq \mu_t$ for all $t \in \text{Im}(\mu)$. But $\mu_0 = G \in \Omega$ because Ω contains all level n -ary subgroups of μ . Hence there exists $t' \in \text{Im}(\mu)$ such that $\mu_{t'} = G$. It follows that $\mu_\beta \supset \mu_{t'} = G$ so that $\mu_\beta = \mu_{t'} = G$ because every level n -ary subgroup of μ is an n -ary subgroup of G .

Now it sufficient to show that $\beta = t'$. If $\beta < t'$, then there exists $t'' \in \text{Im}(\mu)$ such that $\beta \leq t'' < t'$. This implies $\mu_{t''} \supset \mu_{t'} = G$, which is a contradiction. Therefore $\beta = t' \in \text{Im}(\mu)$.

In the case $S \subset G$ we consider the fuzzy set μ_S defined by

$$\mu_S(x) = \begin{cases} \alpha & \text{for } x \in S, \\ 0 & \text{for } x \in G \setminus S. \end{cases}$$

From the proof of Proposition 2.5 it follows that μ_S is a fuzzy n -ary subgroup of G .

Let

$$J = \{i \in I : \mu(y) = t_i \text{ for some } y \in S\}$$

and $\Omega_S = \{(\mu_S)_{t_i} : i \in J\}$. Noticing that Ω_S contains all level n -ary subgroups of μ_S , then there exists $x_0 \in S$ such that $\mu(x_0) = \inf\{\mu_S(x) | x \in S\}$, which implies that $\mu(x_0) = \{\mu(x) | x \in S\}$. This proves that μ attains its infimum on all n -ary subgroups of G .

To prove the converse let μ_α be a nonempty level subset of a fuzzy n -ary subgroup μ . If $\alpha = t$ for some $t \in \text{Im}(\mu)$, then clearly $\mu_\alpha \in \Omega$. If $\alpha \neq t$ for all $t \in \text{Im}(\mu)$, then there does not exist $x \in G$ such that $\mu(x) = \alpha$. But μ_α is nonempty, so there exists $x_0 \in G$ such that $\mu(x_0) > \alpha$. Let $H = \{x \in G : \mu(x) > \alpha\}$. Because $\mu(\overline{x}) \geq \mu(x)$ $x \in H$ implies $\overline{x} \in H$. Moreover for $x_1^n \in H$ we have

$$\mu(f(x_1^n)) \geq \min\{\mu(x_1), \dots, \mu(x_n)\} > \alpha.$$

Hence H is an n -ary subgroup of G . By hypothesis, there exists $y \in H$ such that $\mu(y) = \inf\{\mu(x) | x \in H\}$. But $\mu(y) \in \text{Im}(\mu)$ implies $\mu(y) = t'$ for some $t' \in \text{Im}(\mu)$. Hence $\inf\{\mu(x) | x \in H\} = t' > \alpha$. Note that there does not exist $z \in G$ such that $\alpha \leq \mu(z) < t'$. This gives $\mu_\alpha = \mu_{t'}$. Hence $\mu_\alpha \in \Omega$. Thus Ω contains all level n -ary subgroups of μ . \square

Proposition 3.9. *Let (G, f) be an n -ary group such that every descending chain of its n -ary subgroups terminates at finite step. If μ is a fuzzy n -ary subgroup in G such that a sequence of elements of $\text{Im}(\mu)$ is strictly increasing, then μ has a finite number of values.*

Proof. Assume that $\text{Im}(\mu)$ is not finite. Let $0 \leq t_1 < t_2 < \dots \leq 1$ be a strictly increasing sequence of elements of $\text{Im}(\mu)$. Every level subset μ_{t_i} is an n -ary subgroup of G . For $x \in \mu_{t_i}$ we have $\mu(x) \geq t_i > t_{i-1}$, which implies $x \in \mu_{t_{i-1}}$. Thus $\mu_{t_i} \subseteq \mu_{t_{i-1}}$. But for $t_{i-1} \in \text{Im}(\mu)$ there exists $x_{i-1} \in G$ such that $\mu(x_{i-1}) = t_{i-1}$. This gives $x_{i-1} \in \mu_{t_{i-1}}$ and $x_{i-1} \notin \mu_{t_i}$. Hence $\mu_{t_i} \subset \mu_{t_{i-1}}$, and so we obtain a strictly descending chain $\mu_{t_1} \supset \mu_{t_2} \supset \mu_{t_3} \supset \dots$ of n -ary subgroups, which is not terminating. This contradiction completes the proof. \square

Proposition 3.10. *If every fuzzy n -ary subgroup of G has the finite image, then every descending chain of n -ary subgroup of G terminates at finite step.*

Proof. Suppose there exists a strictly descending chain

$$G = S_0 \supset S_1 \supset S_2 \supset \dots$$

of n -ary subgroups of G which does not terminate at finite step. We prove that μ defined by

$$\mu(x) = \begin{cases} \frac{k}{k+1} & \text{for } x \in S_k \setminus S_{k+1}, \\ 1 & \text{for } x \in \bigcap S_k, \end{cases}$$

where $k = 0, 1, 2, \dots$, is a fuzzy n -ary subgroup with an infinite number of values.

If $f(x_1^n) \in \bigcap S_k$, then obviously $\mu(f(x_1^n)) = 1 \geq \min\{\mu(x_1), \dots, \mu(x_n)\}$.

If $f(x_1^n) \notin \bigcap S_k$, then $f(x_1^n) \in S_p \setminus S_{p+1}$ for some $p \geq 0$. Since $x_1^n \in \bigcap S_k$ implies $f(x_1^n) \in \bigcap S_k$, at least one of x_1, \dots, x_n is not in $\bigcap S_k$. Let S_m be the smallest S_k containing all these elements. For $m > p$ we have $f(x_1^n) \in S_m \subseteq S_{p+1}$, which contradicts to the assumption on $f(x_1^n)$. So, $m \leq p$ and consequently

$$\mu(f(x_1^n)) = \frac{p}{p+1} \geq \frac{m}{m+1} = \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

This proves that μ satisfies the first condition of the Definition 2.3. The second condition is obvious.

Hence μ is a fuzzy n -ary subgroup with an infinite number of different values. Obtained contradiction completes our proof. \square

Proposition 3.11. *Every ascending chain of n -ary subgroups of an n -ary group G terminates at finite step if and only if the set of values of any fuzzy n -ary subgroup of G is a well-ordered subset of $[0, 1]$.*

Proof. If the set of values of a fuzzy n -ary subgroup μ is not well-ordered, then there exists a strictly decreasing sequence $\{t_n\}$ such that $t_n = \mu(x_n)$ for some $x_n \in G$. But in this case n -ary subgroups $B_n = \{x \in G \mid \mu(x) \geq t_n\}$ form a strictly ascending chain, which is a contradiction.

To prove the converse suppose that there exist a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \dots$ of n -ary subgroups. Then $S = \bigcup_{n \in \mathbb{N}} A_n$ is an n -ary subgroup of G and μ defined by

$$\mu(x) = \begin{cases} 0 & \text{for } x \notin S, \\ \frac{1}{k} & \text{where } k = \min\{n \in \mathbb{N} \mid x \in A_n\} \end{cases}$$

is a fuzzy set on G .

We prove that μ is a fuzzy n -ary subgroup. The case when one of x_1, \dots, x_n is not in S is obvious. If all these elements are in S then also $f(x_1^n) \in S$. Let k, m be smallest numbers such that $x_1^n \in A_k$ and $f(x_1^n) \in A_m$. Then $k \geq m$ and

$$\mu(f(x_1^n)) = \frac{1}{m} \geq \frac{1}{k} = \min\{\mu(x_1), \dots, \mu(x_n)\}.$$

So, μ satisfies the first condition of the Definition 2.3. The second condition is obvious.

This means that μ is a fuzzy n -ary subgroup. Since the chain $A_1 \subset A_2 \subset A_3 \subset \dots$ is not terminating, μ has a strictly descending sequence of values. Obtained contradiction proves that the set of values of any fuzzy n -ary subgroup is well-ordered. The proof is complete. \square

Let φ be any mapping from an n -ary group G_1 to an n -ary group G_2 , and μ and λ be fuzzy sets in G_1 and G_2 respectively. Then the *image* $\varphi(\mu)$ and *pre-image* $\varphi^{-1}(\lambda)$ of μ and λ respectively, are the fuzzy sets defined as follows:

$$\varphi(\mu)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)} \{\mu(x)\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset, \end{cases}$$

$$\varphi^{-1}(\lambda)(x) = \lambda(\varphi(x))$$

for all $x \in G_1$ and $y \in G_2$. If φ is a homomorphism then $\varphi(\mu)$ is called the *homomorphic image of μ under φ* .

Proposition 3.12. *Let φ be any mapping from an n -ary group G_1 to an n -ary group G_2 , and let μ be any fuzzy n -ary subgroup of G_1 . Then for $t \in (0, 1]$ we have*

$$\varphi(\mu)_t = \bigcap_{t > \varepsilon > 0} \varphi(\mu_{t-\varepsilon}).$$

Proof. Suppose that $t \in (0, 1]$ and $y = \varphi(x) \in G_2$. If $y \in \varphi(\mu)_t$ then $\varphi(\mu)(\varphi(x)) = \sup_{x \in \varphi^{-1}(\varphi(x))} \{\mu(x)\} \geq t$. Therefore for every real number $\varepsilon > 0$ there exists $x_0 \in \varphi^{-1}(y)$ such that $\mu(x_0) > t - \varepsilon$. So that for every $\varepsilon > 0$, $y = \varphi(x_0) \in \varphi(\mu_{t-\varepsilon})$, and hence $y \in \bigcap_{t > \varepsilon > 0} \varphi(\mu_{t-\varepsilon})$. Conversely, let $y \in \bigcap_{t > \varepsilon > 0} \varphi(\mu_{t-\varepsilon})$, then for each $\varepsilon > 0$ we have $y \in \varphi(\mu_{t-\varepsilon})$ and so there exists $x_0 \in \mu_{t-\varepsilon}$ such that $y = \varphi(x_0)$. Therefore for each $\varepsilon > 0$ there exists $x_0 \in \varphi^{-1}(y)$ and $\mu(x_0) \geq t - \varepsilon$. Hence $\varphi(\mu)(y) = \sup_{x_i \in \varphi^{-1}(y)} \{\mu(x_i)\} \geq \sup_{t > \varepsilon > 0} \{t - \varepsilon\} = t$. So $y \in \varphi(\mu)_t$, and this completes the proof. \square

Theorem 3.13. *Let φ be any homomorphism from an n -ary group G_1 to an n -ary group G_2 , and let μ be any fuzzy n -ary subgroup of G_1 . Then the homomorphic image $\varphi(\mu)$ is a fuzzy n -ary subgroup of G_2 .*

Proof. By Theorem 2.7, $\varphi(\mu)$ is a fuzzy n -ary subgroup if each nonempty level subset $\varphi(\mu)_t$ is an n -ary subgroup of G_2 . If $t = 0$ then $\varphi(\mu)_t = G_2$ and if $t \in (0, 1]$ then by Proposition 3.12, $\varphi(\mu)_t = \bigcap_{t > \varepsilon > 0} \varphi(\mu_{t-\varepsilon})$. So $\varphi(\mu_{t-\varepsilon})$ is nonempty for each $t > \varepsilon > 0$. Thus $\mu_{t-\varepsilon}$ is a nonempty level subset of μ and by Theorem 2.7 is an n -ary subgroup of G_1 . So, the homomorphic image $\varphi(\mu_{t-\varepsilon})$ is an n -ary subgroup of G_2 . Hence $\varphi(\mu)_t$ being an intersection of a family of n -ary subgroups is also an n -ary subgroup of G_2 . \square

Theorem 3.14. *Let φ be a surjection from an n -ary group G_1 to an n -ary group G_2 , and let μ be a fuzzy n -ary subgroup of G_1 which has the sup-property. If $\{\mu_{t_i} \mid i \in I\}$ is the collection of all level n -ary subgroups of μ , then $\{\varphi(\mu_{t_i}) \mid i \in I\}$ is the collection of all level n -ary subgroups of $\varphi(\mu)$.*

Proof. Let $t \in [0, 1]$, then

$$u \in \varphi(\mu) \implies \varphi(\mu)(u) \geq t \implies \sup\{\mu(x) \mid x \in \varphi^{-1}(u)\} \geq t.$$

Since μ has sup-property, this implies that $\mu(x_0) \geq t$ for some $x_0 \in \varphi^{-1}(u)$. Then $x_0 \in \mu_t$ and hence $\varphi(x_0) = u \in \varphi(\mu_t)$. Therefore, we have $\varphi(\mu)_t \subseteq \varphi(\mu_t)$. Now, if $u \in \varphi(\mu_t)$ then $u = \varphi(x)$ for some $x \notin \mu_t$ and hence

$$\varphi(\mu)(u) = \sup\{\mu(z) \mid z \in \varphi^{-1}(u)\} = \sup\{\mu(z) \mid z \in \varphi(z) = \varphi(x)\} \geq \mu(x) \geq t.$$

Therefore $u \in \varphi(\mu)_t$ and hence $\varphi(\mu_t) \subseteq \varphi(\mu)_t$. Thus we have $\varphi(\mu)_t = \varphi(\mu_t)$ for every $t \in [0, 1]$. In particular, $\varphi(\mu)_{t_i} = \varphi(\mu_{t_i})$ for all $i \in I$. Hence all subsets $\varphi(\mu_{t_i})$ are level n -ary subgroups of $\varphi(\mu)$. Also these are the only level n -ary subgroups of $\varphi(\mu)$. \square

The following example shows that surjectiveness of φ in Theorem 3.14 is essential.

Example 3.15. Let (\mathbb{Z}_2, f) and (\mathbb{Z}_4, g) be two ternary groups derived from the additive groups \mathbb{Z}_2 and \mathbb{Z}_4 , respectively. Define $\varphi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ by $\varphi(x) = x$ for $x \in \mathbb{Z}_2$. Then φ is not a surjective homomorphism. Define $\mu : \mathbb{Z}_2 \longrightarrow [0, 1]$ by $\mu(0) = 0.3$ and $\mu(1) = 0.1$. Then μ is a fuzzy ternary subgroup of (\mathbb{Z}_2, f) having sup-property. The level ternary subgroups of μ are $\mu_{0.3} = \{0\}$ and $\mu_{0.1} = G_1$. Now, $\varphi(\mu)$ is defined by

$$\varphi(\mu)(0) = 0.3, \quad \varphi(\mu)(1) = \varphi(\mu)(3) = 0, \quad \varphi(\mu)(2) = 0.1.$$

Hence the level ternary subgroups of $\varphi(\mu)$ are $\{0\}$, $\{0, 2\}$ and $\{0, 1, 2, 3\}$. Therefore $\{\varphi(\mu_{0.3}), \varphi(\mu_{0.1})\}$ does not contain all level ternary subgroups of $\varphi(\mu)$.

4. NORMAL FUZZY n -ARY SUBGROUPS

Definition 4.1. Let μ be a fuzzy set of G . An element $\theta \in G$ is called μ -maximal if $\mu(\theta) \geq \mu(x)$ for all $x \in G$. A fuzzy set μ with the property $\mu(\theta) = 1$ is called *normal*.

Any fuzzy set μ with finite image has a μ -maximal element. In n -ary groups derived from binary group (G, \circ) the identity of (G, \circ) is a μ -maximal element for any fuzzy subgroup of (G, \circ) . In unipotent n -ary groups the element $\theta = f(x_1^n)$ is μ -maximal for all fuzzy n -ary subgroups. Thus a fuzzy n -ary subgroup μ of a unipotent n -ary group is normal if and only if $\mu(\theta) = 1$. Obviously a characteristic function χ_A of any n -ary subgroup A of G is normal.

Proposition 4.2. *Let θ be a μ -maximal element of a fuzzy n -ary subgroup of an n -ary group (G, f) . Then a fuzzy set μ^+ defined by $\mu^+(x) = \mu(x) + 1 - \mu(\theta)$ for all $x \in G$, is a normal fuzzy n -ary subgroup of G which contains μ .*

Proof. Indeed,

$$\begin{aligned} \mu^+(f(x_1^n)) &= \mu(f(x_1^n)) + 1 - \mu(\theta) \geq \min\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} + 1 - \mu(\theta) \\ &= \min\{\mu(x_1) + 1 - \mu(\theta), \mu(x_2) + 1 - \mu(\theta), \dots, \mu(x_n) + 1 - \mu(\theta)\} \\ &= \min\{\mu^+(x_1), \mu^+(x_2), \dots, \mu^+(x_n)\} \end{aligned}$$

and

$$\mu^+(\bar{x}) = \mu(\bar{x}) + 1 - \mu(\theta) \geq \mu(x) + 1 - \mu(\theta) = \mu^+(x).$$

Clearly μ^+ is normal and $\mu \subseteq \mu^+$. \square

It is clear that in a unipotent n -ary group a fuzzy set μ is normal if and only if $\mu^+ = \mu$.

Corollary 4.3. *Let μ and μ^+ be as in the above Proposition. If there is $x \in G$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Proposition 4.4. *If a fuzzy n -ary subgroup μ of an n -ary group has a μ -maximal element, then $(\mu^+)^+ = \mu^+$. Moreover if μ is normal, then $(\mu^+)^+ = \mu$.*

Proof. Straightforward. \square

Proposition 4.5. *Let μ be a fuzzy n -ary subgroup of an n -ary group G . If there exists a fuzzy n -ary subgroup ν of G such that $\nu^+ \subseteq \mu$, then μ is normal.*

Proof. Indeed, for $\nu^+ \subseteq \mu$ we have $1 = \nu^+(\theta) \leq \mu(\theta)$. Hence $\mu(\theta) = 1$. \square

Denote by $\mathcal{N}(G)$ the set of all normal fuzzy n -ary subgroups of G . Note that $\mathcal{N}(G)$ is a poset under the set inclusion.

Proposition 4.6. *Let μ be a non-constant fuzzy n -ary subgroup of an n -ary group G . If μ is a maximal element of $(\mathcal{N}(G), \subseteq)$, then μ takes only the values 0 and 1.*

Proof. Observe that $\mu(\theta) = 1$ since μ is normal. Let $x \in G$ be such that $\mu(x) \neq 1$. We claim that $\mu(x) = 0$. If not, then there exists $a \in G$ such that $0 < \mu(a) < 1$. Let ν be a fuzzy set in G defined by $\nu(x) = \frac{1}{2}(\mu(x) + \mu(a))$ for all $x \in G$. Then clearly ν is well-defined, and

$$\nu(\bar{x}) = \frac{1}{2}(\mu(\bar{x}) + \mu(a)) \geq \frac{1}{2}(\mu(x) + \mu(a)) = \nu(x)$$

for all $x \in G$. Moreover, for all $x_1^n \in G$ we get

$$\begin{aligned} \nu(f(x_1^n)) &= \frac{1}{2}(\mu(f(x_1^n)) + \mu(a)) \geq \frac{1}{2}(\min\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} + \mu(a)) \\ &= \min\{\frac{1}{2}(\mu(x_1) + \mu(a)), \frac{1}{2}(\mu(x_2) + \mu(a)), \dots, \frac{1}{2}(\mu(x_n) + \mu(a))\} \\ &= \min\{\nu(x_1), \nu(x_2), \dots, \nu(x_n)\}. \end{aligned}$$

Hence ν is a fuzzy n -ary subgroup of G . It follows from Proposition 4.2 that $\nu^+ \in \mathcal{N}(G)$ where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(\theta)$ for all $x \in G$. Clearly $\nu^+(x) \geq \mu(x)$ for all $x \in G$. Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(\theta) = \frac{1}{2}(\mu(a) + \mu(a)) + 1 - \frac{1}{2}(\mu(\theta) + \mu(a)) \\ &= \frac{1}{2}(\mu(a) + 1) > \mu(a) \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(\theta)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(G)$. This is a contradiction. \square

We construct a new fuzzy n -ary subgroup from old. Let $t > 0$ be a real number. If $\alpha \in [0, 1]$, α^t shall mean the positive root in case $t < 1$. We define $\mu^t : G \rightarrow [0, 1]$ by $\mu^t(x) = (\mu(x))^t$ for all $x \in G$.

Proposition 4.7. *If μ is a fuzzy n -ary subgroup of an n -ary group G , then so is μ^t . Moreover, if θ is μ -maximal, then $G_{\mu^t} = G_\mu$, where $G_\mu = \{x \in G \mid \mu(x) = \mu(\theta)\}$.*

Proof. For any $x, x_1^n \in G$, we have $\mu^t(\bar{x}) = (\mu(\bar{x}))^t \geq (\mu(x))^t = \mu^t(x)$ and

$$\begin{aligned} \mu^t(f(x_1^n)) &= (\mu(f(x_1^n)))^t \geq (\min\{\mu(x_1), \dots, \mu(x_n)\})^t \\ &= \min\{(\mu(x_1))^t, \dots, (\mu(x_n))^t\} = \min\{\mu^t(x_1), \dots, \mu^t(x_n)\}. \end{aligned}$$

Hence μ^t is a fuzzy n -ary subgroup. Moreover

$$\begin{aligned} G_\mu &= \{x \in G \mid \mu(x) = \mu(\theta)\} = \{x \in G \mid (\mu(x))^t = (\mu(\theta))^t\} \\ &= \{x \in G \mid \mu^t(x) = \mu^t(\theta)\} = G_{\mu^t}. \end{aligned}$$

This completes the proof. \square

Corollary 4.8. *If $\mu \in \mathcal{N}(G)$, then so is μ^t .*

Definition 4.9. A fuzzy set μ defined on G is called *maximal* if it is non-constant and μ^+ is a maximal element of the poset $(\mathcal{N}(G), \subseteq)$.

Proposition 4.10. *If μ is a maximal fuzzy n -ary subgroup of an n -ary group G , then*

- (i) μ is normal,
- (ii) μ takes only the values 0 and 1,
- (iii) G_μ is a maximal n -ary subgroup of G .

Proof. Let μ be a maximal fuzzy n -ary subgroup of G . Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(G), \subseteq)$. It follows from Proposition 4.6 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(\theta)$, and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(\theta) - 1$. By Corollary 4.3, we have $\mu(x) = 0$, that is, $\mu(\theta) = 1$. Hence μ is normal, and clearly $\mu^+ = \mu$. This proves (i) and (ii).

(iii) G_μ is a proper n -ary subgroup because μ is non-constant. Let S be an n -ary subgroup of G containing G_μ . Noticing that, for any subsets A and B of G , $A \subseteq B$ if and only if $\mu_A \subseteq \mu_B$, then we obtain $\mu = \mu_{G_\mu} \subseteq \mu_S$. Since μ and μ_S are normal and $\mu = \mu^+$ is a maximal element of $\mathcal{N}(G)$, we have that either $\mu = \mu_S$ or $\mu_S = \mathbf{1}$ where $\mathbf{1} : G \rightarrow [0, 1]$ is a fuzzy set defined by $\mathbf{1}(x) = 1$ for all $x \in G$. The later case implies that $S = G$. If $\mu = \mu_S$, then $G_\mu = G_{\mu_S} = S$. This proves that G_μ is a maximal n -ary subgroup of G . \square

Definition 4.11. A normal fuzzy n -ary subgroup μ of G is called *completely normal* if there exists $x \in G$ such that $\mu(x) = 0$. The set of all completely normal fuzzy n -ary subgroups of G is denoted by $\mathcal{C}(G)$.

It is clear that $\mathcal{C}(G) \subseteq \mathcal{N}(G)$. The restriction of the partial ordering \subseteq of $\mathcal{N}(G)$ gives a partial ordering of $\mathcal{C}(G)$.

Proposition 4.12. *Any non-constant maximal element of $(\mathcal{N}(G), \subseteq)$ is also a maximal element of $(\mathcal{C}(G), \subseteq)$.*

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(G), \subseteq)$. By Proposition 4.6, μ takes only the values 0 and 1, and so $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$. Hence $\mu \in \mathcal{C}(G)$. Assume that there exists $\nu \in \mathcal{C}(G)$ such that $\mu \subseteq \nu$. Obviously $\mu \subseteq \nu$ also in $\mathcal{N}(G)$. Since μ is maximal in $(\mathcal{N}(G), \subseteq)$ and ν is non-constant, therefore $\mu = \nu$. Thus μ is maximal element of $(\mathcal{C}(G), \subseteq)$. \square

Proposition 4.13. *Maximal fuzzy n -ary subgroup is completely normal.*

Proof. Let μ be a maximal fuzzy n -ary subgroup. By Proposition 4.10 μ is normal and $\mu = \mu^+$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(\theta) = 1$ and $\mu(x) = 0$ for some $x \in G$, which completes the proof. \square

Proposition 4.14. *Let θ be a μ -maximal element of a fuzzy n -ary subgroup of an n -ary group G . If $\varphi : [0, \mu(\theta)] \rightarrow [0, 1]$ is an increasing function, then a fuzzy set μ_φ defined on G by $\mu_\varphi(x) = \varphi(\mu(x))$ is a fuzzy n -ary subgroup. Moreover, if $\varphi(t) \geq t$ for all $t \leq \mu(\theta)$, then $\mu \subseteq \mu_\varphi$.*

Proof. Since f is increasing, then for all $x, x_1^n \in G$ we have

$$\mu_\varphi(\bar{x}) = \varphi(\mu(\bar{x})) \geq \varphi(\mu x) = \mu_\varphi(x)$$

and

$$\begin{aligned} \mu_\varphi(f(x_1^n)) &= \varphi(\mu(f(x_1^n))) \geq \varphi(\min\{\mu(x_1), \dots, \mu(x_n)\}) \\ &= \min\{\varphi(\mu(x_1)), \dots, \varphi(\mu(x_n))\} \\ &= \min\{\mu_\varphi(x), \dots, \mu_\varphi(y)\}. \end{aligned}$$

This proves that μ_φ is a fuzzy n -ary subgroup.

If $\varphi(t) \geq t$ for all $t \leq \mu(\theta)$, then $\mu(x) \leq \varphi(\mu(x)) = \mu_\varphi(x)$ for all $x \in G$, which implies $\mu \subseteq \mu_\varphi$. \square

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